JOURNAL OF COMBINATORIAL THEORY, Series A 23, 354–358 (1977)

Note

On a Theorem of Ryser

D. A. LEONARD

The Ohio State University, Columbus, Ohio 43210

AND

D. KEENAN

California Institute of Technology, Pasadena, California 91125

Communicated by Richard M. Wilson

Received June 17, 1976

In this paper we generalize a theorem of Ryser [1] about finite geometries having an equal number n of points and lines, when something is known about the number δ of lines that miss certain lines. The result is that either (a) all lines have k points, all points are on k lines, and each line misses δ' other lines or (b) all lines have at least k points, all points are on at most $k+1$ lines and n is bounded by a quadratic function of δ. As the bound which is given by the proof is known to be not best possible, we conjecture that $n \leqslant \delta^2 + 5\delta + 4$ in case (b).

If X is a set of elements (points) and $\mathcal{A}$ is a collection of subsets (blocks or lines) of X, we define:

$$\text{for all } x \in X, \qquad r_x = |\{A \in \mathcal{A}: x \in A\}|;$$

$$\text{for all } A \in \mathcal{A}, \qquad \delta_A = |\{B \in \mathcal{A}: A \cap B = \varnothing\}|;$$

and

$$k = \min_{A \in \mathcal{A}} \{|A|\}.$$

Throughout we will also make use of the duality between blocks and points to write $A \in x$ to mean A is a block through, containing, or on x just as we would write $x \in A$ to mean x is a point on A.

THEOREM 1. *Let X be a set of n points, $\mathcal{A}$ a collection of n blocks on X, and δ a positive integer such that:*

(i) *for all* $A \in \mathcal{Cl}$, $|A| \geqslant 3$;

(ii) *for all* $A, B \in \mathcal{Cl}$, $A \neq B$, $|A \cap B| \leqslant 1$; *and*

(iii) $|A| = k$ *implies* $\delta_A \leqslant \delta \leqslant k - 2$.

Then either

(a) *for all* $x \in X$, $r_x = k$, *for all* $A \in \mathcal{Cl}$, $|A| = k$, *and for all* $A \in \mathcal{Cl}$, $\delta_A = \delta' \leqslant \delta$, *or*

(b) *for all* $x \in X$, $r_x \leqslant k + 1$, *for all* $A \in \mathcal{Cl}$, $|A| \geqslant k$, *and* $n = k^2 + \epsilon$, *where* $k + \epsilon \leqslant 2\delta + 2$ (*which forces* $n \leqslant (2\delta + 2)^2$).

Proof. We first state the three basic counting arguments to be used in the proof. Suppose $x \in X$ and $A, B \in \mathcal{Cl}$ with $\{x\} = A \cap B$.

Counting points on the blocks through x,

$$n = |X| \geqslant 1 + \sum_{D \ni x} (|D| - 1). \tag{1}$$

Counting blocks missing A and blocks through points on A,

$$n = |\mathcal{Cl}| = \delta_A + \left[1 + \sum_{y \in A} (r_y - 1)\right]. \tag{2}$$

And counting blocks missing A or B and blocks intersecting both,

$$n = |\mathcal{Cl}| \leqslant (\delta_A + \delta_B) + [r_x + (|A| - 1)(|B| - 1)]. \tag{3}$$

Now fix $A \in \mathcal{Cl}$ with $|A| = k$, and choose $x \in A$ such that $r_x = \max_{y \in A} \{r_y\}$. If x were on no other blocks of size k, then from (1) and (2)

$$1 + 1 \cdot (k - 1) + (r_x - 1)k \leqslant n \leqslant 1 + \delta_A + k(r_x - 1).$$

So

$$k \leqslant 1 + \delta_A,$$

which contradicts hypothesis (iii). Thus there exists a second block B through x with $|B| = k$.

Now applying (1) and (3), we get

$$1 + r_x(k - 1) \leqslant n \leqslant \delta_A + \delta_B + r_x + (k - 1)^2 \tag{4}$$

so

$$r_x \leqslant k + [(\delta_A + \delta_B)/(k - 2)] \leqslant k + 2$$

since $\delta_A, \delta_B \leqslant k - 2$ by (iii).

Suppose for the moment that $r_y \leqslant k$ for all y on at least one block of size k. If z is a point which is not on any block of size k, then from (1) and (2),

$$1 + r_z \cdot k \leqslant n \leqslant 1 + \delta_A + k(k - 1),$$
$$r_z \leqslant k - 1 + (\delta_A/k) < k.$$

Counting point-block incidences two ways,

$$n \cdot k \leqslant \sum_{A \in \mathcal{O}} |A| = \sum_{x \in X} r_x \leqslant n \cdot k.$$

So equality holds, giving $|A| = k$ for all $A \in \mathcal{O}$. And from (2), $n = 1 + \delta_A + k(k-1)$ for all $A \in \mathcal{O}$, so $\delta_A = \delta' \leqslant \delta$ for all $A \in \mathcal{O}$ and conclusion (a) holds.

Returning to (4), suppose $r_x = k + 2$. This forces $\delta_A = \delta_B = \delta = k - 2$ and

$$1 + (k+2)(k-1) = n = (k-2) + (k-2) + (k+2) + (k-1)^2. \tag{4'}$$

This means all blocks through x have size k; all points must be on some block through x, for all $y \neq x$; $r_y \geqslant 1 + (k-1) = k$, since y is on a block through x of size k; and $|D| \geqslant k + 1$ for all $D \in \mathcal{O}$, D not on x, since no block missing x can miss two blocks on x. Thus if $y \neq x$, $r_y \neq k + 2$, and if $r_y = k + 1$, then from (1) and (4′)

$$1 + 1 \cdot (k-1) + k \cdot k \leqslant n = 1 + (k+2)(k-1),$$

which is a contradiction. So $r_y = k$ for all $y \neq x$. Counting point-block incidences,

$$\begin{aligned}(n-1)k + 1(k-2) = \sum_{x \in X} r_x &= \sum_{A \in \mathcal{O}} |A| \\ &\geqslant (k+2)k + (n-k-2)(k+1).\end{aligned}$$

So

$$\begin{aligned}2k + 4 &\geqslant n = 1 + (k+2)(k-1), \\ k^2 - k - 5 &\leqslant 0.\end{aligned}$$

But from (i), $k \geqslant 3$. So $r_x \neq k + 2$.

We now have $r_y \leqslant k + 1$ for all y on at least one block of size k. If z is not on any block of size k, then from (1) and (2)

$$\begin{aligned}1 + r_z k \leqslant n &\leqslant 1 + \delta + k \cdot k, \\ r_z &\leqslant k + (\delta/k) < k + 1.\end{aligned}$$

So $r_y \leqslant k + 1$ for all $y \in X$. From (1)

$$n \geqslant 1 + (k+1)(k-1) = k^2, \qquad \text{so let} \quad n = k^2 + \epsilon.$$

Then from (3),

$$k^2 + \epsilon = n \leqslant 2\delta + (k+1) + (k-1)^2$$

giving $k + \epsilon \leqslant 2\delta + 2$, and (b) holds, proving the theorem.

The bound obtained in the proof is not best possible. It is conjectured that the best bound is $n \leqslant \delta^2 + 5\delta + 4$, which conforms to the known results of Ryser [1] and Keenan [unpublished] for $\delta = 1$ and $\delta = 2$, respectively. One would hope that the simple geometric construction for the maximal sizes would generalize past $\delta = 2$, and to prove the bound one need only show $\delta_A = k - 2$ for some A with $|A| = k$. It would be nice if these special configurations were self-dual.

The hypothesis that $\delta \leqslant k - 2$ is not completely necessary, so we prove the following.

THEOREM 2. *Replacing* (iii) *in Theorem* 1 *by*

(iii′) $\delta_A \leqslant \delta$ *for all* $A \in \mathcal{A}$ *with* $|A| = k$ *or* $k + 1$.

Then either Theorem 1 *applies or*

(c) $k \leqslant \delta + 1$ *and* $n \leqslant \delta^2 + 5\delta + 2$.

Proof. If $k \geqslant \delta + 2$, then Theorem 1 applies, so we may assume $k \leqslant \delta + 1$. Let $r = \max_{x \in X}\{r_x\}$.

Suppose two blocks of size k intersect. Then as before from (1) and (3)

$$1 + r(k-1) \leqslant n \leqslant 2\delta + r + (k-1)^2$$
$$r \leqslant k + [2\delta/(k-2)] \leqslant k + 2\delta.$$

And since $k \leqslant \delta + 1$,

$$n \leqslant 2\delta + (k + 2\delta) + (k-1)^2,$$
$$n \leqslant 2\delta + (3\delta + 1) + \delta^2 = \delta^2 + 5\delta + 1.$$

If A and B are two blocks of size k which do not intersect, then by (2)

$$n \leqslant \delta_B + (k \cdot k + \delta_A),$$
$$n \leqslant 2\delta + (\delta + 1)^2 = \delta^2 + 4\delta + 1.$$

So we may assume that there is only one block A of size k. Let B be the next smallest block, and let $|B| = k + \alpha$. From (1) and (2),

$$1 + (r-1)(k + \alpha - 1) + 1 \cdot (k-1) \leqslant n \leqslant 1 + \delta + k(r-1),$$
$$(r-1)(\alpha - 1) \leqslant \delta + 1 - k.$$

If $\alpha > 1$

$$r \leqslant \delta + 1 - k \leqslant \delta - 2$$

and

$$n \leqslant 1 + \delta + (\delta + 1)(\delta - 3) = \delta^2 - \delta - 2.$$

So $|B| = k + 1$.

If $A \cap B \neq \varnothing$, then from (1) and (3)

$$1 + (r - 1)k + 1 \cdot (k - 1) \leqslant n \leqslant 2\delta + r + k(k - 1),$$
$$r \leqslant k + [2\delta/(k - 1)] \leqslant k + \delta.$$

So

$$n \leqslant 2\delta + (k + \delta) + k(k - 1),$$
$$n \leqslant 2\delta + (2\delta + 1) + (\delta + 1)\delta = \delta^2 + 5\delta + 1.$$

If $A \cap B = \varnothing$, then from (2)

$$n \leqslant \delta_B + (k(k + 1) + \delta_A),$$
$$n \leqslant 2\delta + (\delta + 1)(\delta + 2) = \delta^2 + 5\delta + 2,$$

which finishes the proof.

Reference

1. H. J. Ryser, Subsets of a finite set that intersect each other in at most one element, *J. Combinatorial Theory A* **17** (1974), 59–74.